Divergence Thm

$D \subseteq \mathbb{R}^n$, compact

$\vec{F} : D \to \mathbb{R}^n$ a vector field

Divergence of $\vec{F}$, $\nabla \cdot \vec{F}$ is defined as:

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}$$

(Gradient of a scalar function $\varphi : D \to \mathbb{R}$:

$$\nabla \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \ldots, \frac{\partial \varphi}{\partial x_n} \right)^T$$
)

Product Rules: $\varphi : D \to \mathbb{R}$ a scalar function

$$\nabla \cdot (\varphi \vec{F}) = (\nabla \varphi) \cdot \vec{F} + \varphi (\nabla \cdot \vec{F})$$

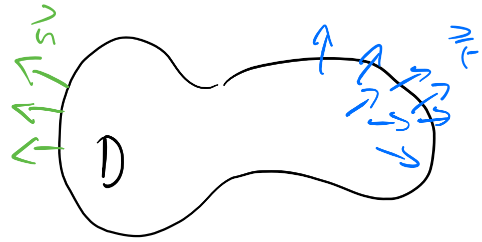
Thm (Divergence Thm)

Then

$$\int_D \nabla \cdot \vec{F} = \int_{\partial D} \vec{F} \cdot \vec{n}$$

where $\vec{n}$: outward unit normal vector

Intuition:



Imagine water is flowing in a region D.

$\vec{F}$ is the velocity of the water flow

$\nabla \cdot \vec{F}(\vec{p})$ means the amount of water flowing out at point $\vec{p}$.

Summing the amount of water flowing out at every points of D, which is $\int_D \nabla \cdot \vec{F}$, equals to the amount of water going outside of the boundary, $\int_{\partial D} \vec{F} \cdot \vec{n}$

Multivariate Integration by Parts

Put $\vec{F} = \varphi \vec{\Gamma}$, φ some scalar function,

$\vec{\Gamma}$ some vector field,

$$\int_D \nabla \cdot \vec{F} = \int_{\partial D} \vec{F} \cdot \vec{n}$$

$$\Rightarrow \int_D \varphi (\nabla \cdot \vec{\Gamma}) + (\nabla \varphi) \cdot \vec{\Gamma} = \int_{\partial D} \varphi (\vec{\Gamma} \cdot \vec{n})$$

Energy Minimization (for denoising)

(See Also Calculus of Variations / Euler - Lagrange Equations)

For an observed noisy image $g : D \subseteq \mathbb{R}^2 \to \mathbb{R}$, where $g \equiv 0$ on ∂D

Want to find clean image $f : D \to \mathbb{R}$ where $f \equiv 0$ on ∂D

by minimizing the energy:

$$E(f) = \underbrace{\int_D (f(x,y) - g(x,y))^2}_{\text{Fidelity term}} + \underbrace{\int |\nabla f|^2}_{\text{Regularity term}}$$

Fidelity term: small when f closed to g

Regularity term: small when f is smooth

Or more Generally,

$$E(f) = \int_D K_1(x,y)(f(x,y) - g(x,y))^2 + \int K_2(x,y) |\nabla f|^2$$

K_1, K_2 : weights for the fidelity and regularity terms.

If K_1 large, K_2 small at some point (x_1, y_1), f will be more closed to g at the point.

If K_1 small, K_2 large at some point (x_2, y_2), f will be more smooth, rather than closed to g.

Suppose f is a minimizer to the energy.

Let $v : D \to \mathbb{R}$, with $v \equiv 0$ on ∂D

(this v is called variation, that's why "Calculus of Variations")

$\varepsilon \in \mathbb{R}$

Define $S(\varepsilon) := \mathcal{E}(f + \varepsilon v)$

$S : \mathbb{R} \to \mathbb{R}$ is a scalar function.

S attains minimum at $\varepsilon = 0$

since f is a minimizer to $\mathcal{E}$,

$$\mathcal{E}(f) \le \mathcal{E}(f + \varepsilon v) \quad \forall \varepsilon$$

Then $\left.\frac{d}{d\varepsilon} S(\varepsilon)\right|_{\varepsilon=0}$ (first derivative Test).

$$\frac{d}{d\varepsilon} S(\varepsilon) = \frac{d}{d\varepsilon} \mathcal{E}(f + \varepsilon v)$$

$$= \frac{d}{d\varepsilon} \iint_D \left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right)^2 dx\,dy$$

$$+ \frac{d}{d\varepsilon} \iint_D \langle \nabla f + \varepsilon \nabla v, \nabla f + \varepsilon \nabla v \rangle \, dx\,dy$$

$$= \langle \nabla f, \nabla f \rangle + 2\varepsilon \langle \nabla f, \nabla v \rangle + \varepsilon^2 \langle \nabla v, \nabla v \rangle$$

$$= \iint_D 2\left(f(x,y) + \varepsilon v(x,y) - g(x,y)\right) v(x,y)\, dx\, dy$$
$$+ \iint_D 2 \langle \nabla f, \nabla v \rangle + 2\varepsilon \langle \nabla v, \nabla v \rangle$$

$$\frac{d}{d\varepsilon} S(\varepsilon) \Big|_{\varepsilon = 0} = 0$$

$$\Rightarrow \quad 0 = \int_D 2\left(f(x,y) - g(x,y)\right) v(x,y)$$
$$+ \int_D 2 \nabla f \cdot \nabla v$$

Recall Integration by Parts:

$$\int_D \varphi (\nabla \cdot \vec{F}) + (\nabla \varphi) \cdot \vec{F} = \int_{\partial D} \varphi (\vec{F} \cdot \vec{n})$$

Put $v = \varphi$, $\nabla f = \vec{F}$,

$$\int \nabla f \cdot \nabla v = \int (\nabla \varphi) \cdot \vec{F}$$
$$= \int_{\partial D} \varphi (\vec{F} \cdot \vec{n}) - \int \varphi (\nabla \cdot \vec{F})$$
$$= \int_{\partial D} \underset{=}{v} (\nabla f \cdot \vec{n}) - \int v (\nabla \cdot (\nabla f))$$

we assume $v \equiv 0$ on ∂D

$$= - \int v (\nabla \cdot \nabla f)$$

$$0 = \int_D \left(f(x,y) - g(x,y) - \Delta f \right) \underline{v(x,y)}$$

arbitrary with
$v \equiv 0$ on ∂D

$$\Rightarrow \begin{cases} f(x,y) - g(x,y) - \Delta f(x,y) = 0 \\ f(x,y) = 0 \quad \text{on } \partial D \end{cases}$$

For the more general model

$$E(f) = \int_D K_1(x,y)\left(f(x,y) - g(x,y)\right)^2 + \int K_2(x,y) |\nabla f|^2,$$

If f is the minimizer to E,

$$S(\varepsilon) := E(f + \varepsilon v)$$

$$\frac{d}{d\varepsilon} S(\varepsilon) \Big|_{\varepsilon=0} = 0$$

$$\frac{d}{d\varepsilon} S(\varepsilon) = \frac{d}{d\varepsilon} \int_D K_1 (f + \varepsilon v - g)^2 + \frac{d}{d\varepsilon} \int_D K_2 |\nabla f + \varepsilon v|^2$$

$$= \int_D 2K_1 (f + \varepsilon v - g) v + \int_D K_2 (2 \nabla f \cdot \nabla v + 2\varepsilon \nabla v \cdot \nabla v)$$

$$\frac{d}{d\varepsilon} S(\varepsilon) \Big|_{\varepsilon=0} = 0$$

$$\Rightarrow \quad 0 = \int K_1 (f - g) v + \int_D K_2 \nabla f \cdot \nabla v$$

Put $\vec{F} = K_2 \nabla f$, $\varphi = v$

$$\int_D \varphi \, \nabla\cdot\vec{F} + \nabla\varphi\cdot\vec{F} = \int_{\partial D} \varphi(\vec{F}\cdot\vec{n})$$

$$\Rightarrow \int_D (K_2 \nabla f)\cdot\nabla v = \underbrace{\int_{\partial D} v(K_2 \nabla f\cdot\vec{n})}_{} = 0 \quad v \equiv 0 \text{ on } \partial D$$

$$- \int v \, \nabla\cdot(K_2 \nabla f)$$

$$0 = \int K_1 (f-g) v + \int_D K_2 \nabla f\cdot\nabla v$$

$$\Rightarrow 0 = \int \Big(K_1(f-g) - \nabla\cdot(K_2 \nabla f) \Big) v$$

$$\Rightarrow \begin{cases} K_1(f-g) - \nabla\cdot(K_2\nabla f) = 0 \\ f \equiv 0 \quad \text{on } \partial D \end{cases}$$

Remark

If we not assume $f, g, v \equiv 0$ on boundary, then the steps for eliminating the boundary terms,

e.g: $\int_{\partial D} v(K_2 \nabla f\cdot\vec{n})$ in the second energy, requiring an extra condition $K_2 \nabla f\cdot\vec{n} = 0$

Then the PDE becomes :

$$\begin{cases} K_1 (f - g) - \nabla \cdot (K_2 \nabla f) = 0 \\ K_2 \nabla f \cdot \vec{n} = 0 \quad \text{on } \partial D \end{cases}$$

This boundary condition calls

Neumann Boundary Condition,

and the boundary condition of :

$$\begin{cases} K_1 (f - g) - \nabla \cdot (K_2 \nabla f) = 0 \\ f \equiv 0 \quad \text{on } \partial D \end{cases}$$

calls Dirichlet Boundary Condition.